Use of Chaos in a Lyapunov Dynamic Game

J. Skowronski
Late, Ex-Professor, Mechanical Engineering, University of Southern California, Los Angeles, CA 90089-1453, U.S.A.
W. J. Grantham
Professor, Mechanical and Materials Engineering, Washington State University, Pullman, WA 99164-2920, U.S.A.
B. Lee*
Assosiate Professor, Department of Mechanical and Automotive Engineering, Keimyung University, Daegu 704-701, Republic of Korea

Feedback strategies of a qualitative competitive game between two players can be designed such as to influence parameters of a mechanical system to induce chaotic behavior. The purpose is to reduce the options and effects of the opponent's strategy. We show it on a case with dynamics specified by a nonautonomous Duffing equation with the players represented by damping and external forcing, respectively. It seems however that the conclusions can be made valid generally.

Key Words : Qualitative Game, Winning Set, Chaos, Duffing's Equation

1. Posing the Problem

As a rule, chaotic behavior of nonlinear mechanical systems is a phenomenon we would like to avoid when designing and using such systems. It can destroy a desired degree of stability or attaining of a control objective, say, reaching source of target by the system trajectories. Looking however from the viewpoint of an opposition to a control program that actuates the motion of the system with a specific objective, the chaotic behavior may prove a usable tool which, if it did not prevent the controlling agent from attaining the said objective, at least it may destroy the precision of doing so (Lee, 1995). We shall make the opposition an active player and discuss the case on the scalar Duffing-type equation (Awrejcewicz, 1988, Ueda, 1980)

$$\ddot{q}+D(q, \dot{q}, u^2)+\Pi(q)=u^1 \tag{1}$$

although our conclusion could easily be augmented to the multi-dimensional state space. In the latter case, eq. (1) becomes a fairly general model of an inertially decoupled mechanical system. Indeed, eq. (1) follows from the Lagrange equation of motion with generalized displacement $q(t)$, $t\geqq t_0$, $t_0\in R$, ranging in the bounded set of constraints Δ_q and generalized velocity $\dot{q}(t)$, $t\geqq t_0$, within the bounded set of velocity constraints $\Delta_{\dot{q}}$. The functions, $D(\cdot)$, $\Pi(\cdot)$ represent internal forces per inertia, successively non-potential (Coriolis, gyro, damping) and potential (restoring), while $u^1(t)$ is the control variable representing an external force per inertia exerted by an actuator with a programmed behavior, shaped towards a stipulated objective. The latter means a desired pattern of solutions to eq. (1) in the phase space $Q_{q\dot{q}}$. Such a pattern may, for instance be reaching of a certain energy level. Obviously the force per inertia u^1 produces a non-zero power thus changing the total energy of

* Corresponding Author,
E-mail : blee@kmu.ac.kr
TEL : +82-53-580-5349; **FAX** : +82-53-580-6285
Assosiate Professor, Department of Mechanical and Automotive Engineering, Keimyung University, Daegu 704-701, Republic of Korea. (Manuscript **Received** April 14, 2003; **Revised** August 20, 2003)

the system. It's natural opponent is located in the internal energy changing (non-potential) force $D(\cdot)$, controlled by the control $u^2(t)$, $t \geqq t_0$. To accommodate viscous damping we shall assume

$$D(q, 0, u^2) = 0, \ \forall q, u^2 \qquad (2)$$

Letting $\nu(q)$ to be the potential energy of the system, by definition of the potential forces are $\Pi(q) = \nabla\nu(q)$. To cover $\nu(q)$ both harmonic and anharmonic including trigonometric functions, we express $\Pi(q)$ in terms of the series

$$\Pi(q) = \alpha + \beta q^3 + \cdots \qquad (3)$$

with the higher power terms truncated unless specifically mentioned. We can now view eq. (1) as the dynamics of a differential game between players 1, 2 each with the objective referring to a specific qualitative behavior of the motion of eq. (1) and each with a control program to be designed in order to attain its objective. The first program is designed as an external excitation sinusoidal in shape with parameters λ, ω to be adjusted

$$\begin{aligned} u^1(t) &= P^1[\lambda(q(t), \dot{q}(t)), t] \\ &= \lambda(q, \dot{q}) \sin \omega t \end{aligned} \qquad (4)$$

The second players program will generally be displacement and velocity dependent:

$$u^2(t) = P^2(q(t), \dot{q}(t)) \qquad (5)$$

reducing the function $D(\cdot)$ to $D(q, \dot{q}, u^2) = \tilde{D}(q, \dot{q})$. The programs are $P^i(\cdot)$ such that $u^i(t)$ range in the set of control constraints $U_i : |u^i(t)| \leq \hat{u}^i = \text{const}$, while λ ranges in the set $\Lambda : 0 \leq \lambda(q, \dot{q}) \leq \tilde{\lambda} = \text{const}$. Unless otherwise stated we will consider the control additive:

$$D(q, \dot{q}, u^2) = d(q, \dot{q}) + u^2 \qquad (6)$$

The total energy of the system forms the surface

$$z = E(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\alpha q^2 + \frac{1}{4}\beta q^4 \qquad (7)$$

over the region of constraints $\Delta = \Delta_q \times \Delta_{\dot{q}}$ in the phase-space R^{2n}, with extrema identified with the equilibria of eq. (1) by definition of $\nu(q)$. The Dirichlet stable equilibria correspond to the minima and their neighborhood wells $\Delta_E \subset \Delta$ are determined by the restituting condition

$$\Delta_E : \Pi(q) q \geqq 0 \qquad (8)$$

provided the origin (0, 0) has been moved to the equilibrium concerned, see Fig. 1. Each two wells are separated by the energy threshold where local maximum of $\nu(q)$ interfaces with the corresponding minimum of kinetic energy $T = \frac{1}{2}\dot{q}^2$. The threshold thus correspond to saddle point (unstable) equilibria in Δ. The energy levels

$$E(q, \dot{q}) = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\alpha q^2 + \frac{1}{4}\beta q^4 = \text{const} \qquad (9)$$

form the first integral of the conservative subsystem of eq. (1) which serves as a reference frame for the motions of eq. (1) on Δ. The thresholds are the levels passing through the unstable equilibrium points ($q = q^e$, $\dot{q} = 0$) in Δ, with the lower of the two that enclose each well, defining this well i.e., enclosing the set of points $(q, \dot{q})$ specified by eq. (8). Within a well, at each of the levels eq. (threshold inclusive) we have points of entry eq. (direction of motion against $+\nabla E$ or exit (opposite direction) of the motions of eq. (1), depending on whether the power of the energy changing forces

$$\dot{E}(q, \dot{q}) = u^1\dot{q} - [d(q, \dot{q}) + u^2]\dot{q} \qquad (10)$$

balances negatively or positively. For sliding on the level (points of contact) $E(q, \dot{q}) = \text{const}$ and the power vanishes.

Suppose that the external forcing power is limited to the extent of confining the motions below

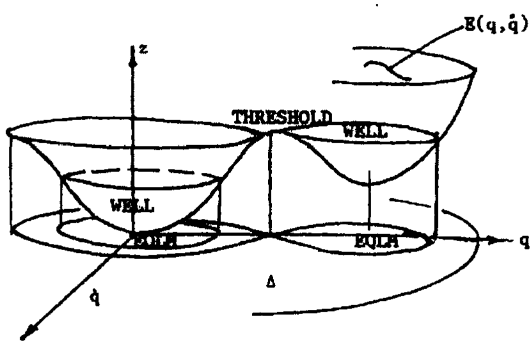

Fig. 1 Two energy wells separated by threshold

the threshold of a local well, with periodic forcing it is plausible for player 1 to aim at controlling the motions of eq. (1) towards a target limit cycle just inside the threshold. Then, if he can make the attractor chaotic and fractal, there is a chance of pushing the motions above the threshold, out of the well. To achieve his objective he would have to choose suitable values of his coefficients λ, ω. Obviously his opponent player 2 may use positive damping $D(q, \dot{q}, u^2)\dot{q} \geqq 0$ in eq. (10), to push the motions down the energy levels towards the equilibrium, which is his target. There will be regions in Δ_E, possibly close to the equilibrium where player 2 may succeed and regions where he may not, possibly for larger amplitudes. These are called the winning regions for corresponding players. For instance, for the case of $\alpha=1$, $\beta=-0.6$, $d=\dot{q}$ and the programs determined by $\lambda=0.1185$, $\omega=0.555$, $u^2=0.6\dot{q}$, Thomspon-Stewart (1986) show that in the basic well there coexists two limit cycles about (0, 0): a single valued closed E-level at lower amplitudes attainable from small neighborhood of (0, 0) which is thus our winning set for player 2 and another, chaotic attractor in which the peak amplitude varies from one cycle to another and may even jump over the threshold due to the fractal structure which gives folding points along the q-axis (see Fig. 2). Such attractor is obviously reachable from another region of larger amplitudes which is winning for the player 1 and uncontrollable for player 2.

In the above, the player 1 actually uses chaos to attain his objective. Obviously between the two winning sets we should have some separating barrier surface which, if found, makes it possible to estimate or even determine such winning sets, thus specifying what is usually called the map of the game in Δ for the objectives concerned. The qualitative game is solved when such a map is found.

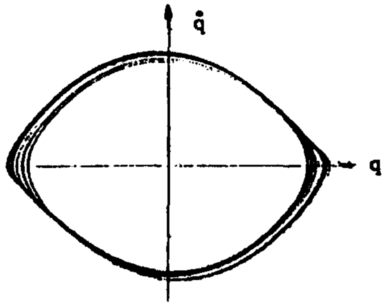

Fig. 2 An chaotic attractor

The roles of the players 1, 2 may be inverted, i.e., player 1 may find $P^1(\cdot)$ such that, by eq. (10), he can generate a single valued limit cycle at some level about (0, 0) while the player 2 uses negative damping generating self-sustained oscillations opposing the task and forcing the motions above the threshold i.e., out of the well Δ_E.

The two objectives described are just one of many pairs of qualitative objectives possible in applications of nonlinear models of mechanical systems, to which qualitative differential games may be used. We consider our case a preliminary study of such usage.

We need now certain facts from the qualitative game theory to specify conditions for solving our problem.

2. The Game Theoretic Discussion

Let us write eq. (1) in the state format substituting $\bar{x}(t)=(x_1(t), x_2(t))^T$, $x_1=q$, $x_2=\dot{q}$, and $\bar{f}=(f_1, f_2)^T$, $f_1=\dot{q}$, $f_2=-D(\bar{x}, u^2)-\Pi(x_1)+u^1$. The philosophy of the qualitative game requires determining strategies i.e., feedback control programs for the players such that each objective is achieved no matter what admissible strategy is used by the opposition. More formally it thus requires an interface between two semigames each expressed by the contingent equation

$$\dot{\bar{x}} \in \{\bar{f}(\bar{x}, t, y^1, u^2) \mid u^i\} = P^i(\bar{x}, t) \qquad u^1 \in U_j,\ i, j=1, 2,\ i \neq j \tag{11}$$

We call eq. (11) the semigame for player i, briefly the i-game meaning that player i is active and playing against all options of player j, $j \neq i$, expressed by the compact control constraint set U_j, $j=2, 1$, (see Section 1). For suitable functions $\bar{f}(\cdot)$, $P^i(\cdot)$, by Filippov (1977), the eq. (11) has through each $\bar{x}^0=\bar{x}(t_0) \in \Delta$ absolutely continuous solutions $\phi(x_0, t_0, \cdot) : R \mapsto \Delta$ and conversely there is a pair $u^1(\cdot)$, $u^2(\cdot)$ such that the corresponding $\phi(\cdot)$ satisfies the so called selector equation: $\dot{\phi}(t)=\bar{f}(\phi(t),\ u^1,\ u^2,\ t)$, $\forall t \geqq t_0$. We shall denote the class of such solutions by $X^i(\bar{x}^0,\ t_0)$.

Following our discussion in section 1, the objective of the i-game is the capture of the motions of eq. (1) in the target T^i which is a given compact set in Δ:

$$\begin{aligned} T^1&=\{(x_1,\ x_2)\in\triangle \mid E(x_1,\ x_2)\geqq c^1\} \\ T^2&=\{(x_1,\ x_2)\in\triangle \mid E(x_1,\ x_2)\leqq c^2\} \end{aligned} \tag{12}$$

where c^2 is a small positive constant and $c^1>0$ represents the energy threshold: $c^1=\frac{1}{4}\alpha^2/\beta$.

Definition 1 *The game is strongly i-controllable at $(\bar{x}^0,\ t_0)\in\triangle\times R$ for capture in T^1 if there is a strategy (program) $P^1(\cdot)$ and a constant $T_c\geqq 0$ such that $\phi(\bar{x}^0,\ t_0,\ \cdot)\in X^i(\bar{x}^0,\ t_0)$ implies*

$$\phi(\bar{x}^0,\ t_0,\ t)\in T^i,\ \forall t\geqq t_0+T_c$$

Let $\bar{x}^0(R)$ denote the t_0-family of points $(\bar{x}^0,\ t_0)$, $t_0\in R$. The set of all points $\bar{x}^0$ in Δ satisfying Definition 1 forms the region of strong i-controllability for capture in T^i, denoted Δ_c^i, and any subset of Δ_c^i is strongly i-controllable for capture. It is obvious that we must have $T^i\cap\Delta_c^i\neq\emptyset$ and that by definition, Δ_c^i is strongly positively invariant under $P^i(\cdot)$: $\bar{x}^0\in\Delta_c^i\Rightarrow\phi(\bar{x}^0,\ t_0,\ R^+)\subset\Delta_c^i$. Then also it follows that there is a nonempty subset T_c^i of $T^i\cap\Delta_c^i$ which is positively strongly invariant under $P^i(\cdot)$ and is called a capturing subtarget. There may be many such subtargets, but if a chaotic attractor exists, it must belong to one of them.

Let Δ_0^i, T_c^i be two subsets of Δ such that $\Delta_0^i\cap T_c^i\neq\emptyset$ and let $V^i(\cdot)$: $\Delta\mapsto R$ be a C^1-function defining T_c^i :

$$\partial T_c^i\ :\ V^i(\bar{x})=\text{const}=v_\tau^i \tag{13}$$

and such that

$$v_0^i=\inf V^i(\bar{x})\mid\bar{x}\in\partial\Delta_0^i>v_\tau^i \tag{14}$$

Introduce $CT_c^i=\Delta_0^i-T_c^i$ and open $S^i\subset\overline{CT_c}$ such that $S\cup\{0\}=\emptyset$. The following theorem can be proved by the same argument as used for the capture conditions in (Skowronski, 1988).

Theorem 1 *The set Δ_0^i is strongly i-controllable for capture in T^i within T_c^i if there are two functions: $P^i(\cdot)$ on CT_c^i, and $V^i(\cdot)$: $S^i\mapsto R$, such that*

(i) $V^i(\bar{x})\leqq v_0^i\bar{x}\in CT_c^i$
(ii) $V^i(\bar{x})\leqq v_0^i\bar{x}\in S^i\cap T_c^i$
(iii) *or each $u^i=P^i(\bar{x},\ t)$ there is $T_c^i<\infty$ such that*

$$\nabla V^i(\bar{x})^T\bar{f}(\bar{x},\ t,\ u^1,\ u^2)\leqq-\frac{v_0^1-v_\tau^i}{T_c^i} \tag{15}$$

for all $u^j\in U_j$, $j\neq i$.

The program P^i may be calculated from the immediate corollary:

Corollary 1 *Given $\bar{x}\in CT_c^i$, if there is a pair $\tilde{u}^1,\ \tilde{u}^2\in U_1\times U_2$ such that*

$$\begin{aligned} &\Delta V^i(\bar{x})^T\bar{f}(\bar{x},\ t,\ \tilde{u}^1,\ \tilde{u}^2) \\ &=\min_{u^i}\max_{u^j}(\nabla V^i(\bar{x})^T f(\bar{x},\ t,\ u^1,\ u^2)) \\ &\leq-\frac{v_0^i-v_\tau^i}{T_c^i} \end{aligned} \tag{16}$$

then condition (iii) *of Theorem 1 is met with $\tilde{u}^i=\tilde{P}^i(\bar{x},\ t)$.*

The second corollary has been proved in (Skowronski, 1986).

Corollary 2 *If the boundary $\partial\Delta_0^i$ is defined as a V^i-level:*

$$\partial\Delta_0^i\ :\ V^i(\bar{x})=\text{const}=v_0^i>v_\tau^i$$

given $V^i(\cdot)$, $P^i(\cdot)$, condition (iii) *is also necessary for Δ_0^i being strongly controllable for capture in T^i within T_c^i.*

If so, then measuring the distance from ∂T_c^i in terms of a specified norm $V^i(\bar{x})$ we can postulate that, given $P^i(\cdot)$, v_0^i and thus also $\partial\Delta_0^i$ can be pushed away from ∂T_0^i as far until eq. (15) is contradicted. The latter is obviously qualified by the choice of $V^i(\cdot)$ and for an arbitrary $V^i(\cdot)$, this is as far as we can go. On the other hand Δ_c^i is a maximal controllable Δ_0^i or a union of such sets in Δ (see Skowronski, 1989).

How far does Δ_c^i stretch? Introduce the semi-neutral set $\Delta_N^i=\Delta-\Delta_c^i$ covered by points where Definition 1 is contradicted. Then introduce a surface Σ^i subdividing Δ into two disjoint sets $\Delta^i\supset\Delta_c^i$ called interior and $C\Delta^i=\Delta-\Delta^i$ called exterior, with the property that for $\bar{x}^0\in\Sigma^i$ there is $P^j(\cdot)$, $j\neq i$ such that $\phi(\bar{x}^0, t_0, R^+)\cap\Delta^i=\emptyset$ for all motions of $X^i(\bar{x}, t_0)$. Such Σ^i is designated nonpermeable for player i, briefly i-nonpermeable.

Theorem 2 *A surface S partitioning Δ is Σ^i if there are $P^i(\cdot)$ and a C^1-function $V_B^i(\cdot): D\mapsto R$, D(open)$\supset S$, such that for all $\bar{x}^0\in\Delta^i$.*

(i) $V_B^i(\bar{x})<V_B^i(\hat{x}),\ \hat{x}\in S$
(ii) given $u^j=P^j(\bar{x}, t)$

$$\nabla V_B^i(\bar{x})^T\bar{f}(\bar{x}, t^1, u^1, u^2)\geq 0,\ \forall u^i\in U_i \quad (17)$$

The theorem was proved in (Skowronski, 1986).

Quite naturally $\Sigma^i\subset\Delta_N^i$ and there may be many of them, but only one that is closest to Δ_c^i. If $\partial\Delta_c^i$ is not defined, which is the case when we search for it, then we choose Σ^i closest to the target T^i (Skowronski, 1987; Arderna, 1989) and call it i-semibarrier B^i. It is convenient if B^i is the boundary $\partial\Delta_c^i$.

Then we may introduce the neutral zone $\Delta_N=\Delta-(\Delta_c^1\cup\Delta_c^2)$, closed if both Δ_c^i are open, but possibly empty, and define the barrier $B=B^1\cap B^2$, obviously in Δ_N, and separating Δ_c^1, Δ_c^2 if not empty. The barrier does not necessarily partition Δ. Since $B^i\subset\Delta_N^i$, $i=1, 2$ wherefrom $B\subset\Delta_N$. From the definition of B we conclude immediately that it is *nonpermeable for both players* and *unique* i.e., there is only one such surface between Δ_c^1 and Δ_c^2.

Any candidate for B, whether provided by necessary conditions such as dynamic programming, Isaac's barrier, ... etc., or obtained from an educated guess of a practitioner, may be confirmed by using Theorem 2 twice, successively for each player $i=1, 2$.

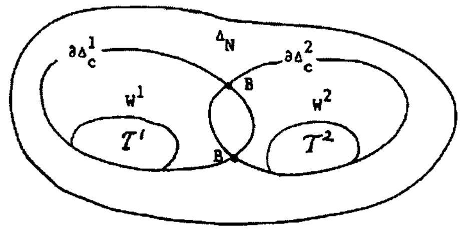

Fig. 3 Hypothetical map of the game

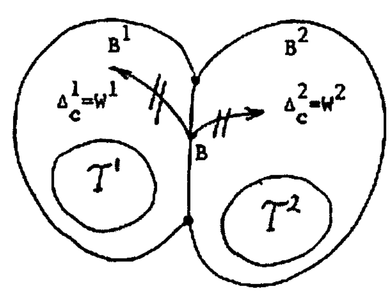

Fig. 4 Disjoining regions

In general, the regions Δ_c^1 may not be disjoint i.e., $\Delta_c^1\cap\Delta_c^2\neq\emptyset$. The winning set for the player i is defined as $W^i=\Delta_c^i-(\Delta_c^1\cap\Delta_c^2)$, $i=1, 2$. The complements of the union $\Delta_c^1\cup\Delta_c^2$ to Δ are filled up by the so called draw regions, which again can be classified (Skowronski, 1987). We may now draw a hypothetical map of the game shown in Fig. 3 with the barrier reduced to two points, or an alternative local picture with disjoining regions $\Delta_c^i=W^i$, $i=1, 2$, shown in Fig. 4.

3. Mechanical Systems

We may now return to the motion equation eq. (1), with the energy in eq. (7), the power in eq. (10) and the objectives of capture discussed in Section 1. Consequently to the latter, we shall consider the targets in eq. (12) with the capturing subtargets proposed as $T_c^i=T^i$, $i=1, 2$ of eq. (12), both enclosed in the same energy well (see Fig. 5). Note that the target T^1 leaves the option of the motions escaping Δ_E and thus tending to some limit cycle in the next well. Our discussion in this section follows the results by Skowronski (1989).

Choosing $V^i(\cdot)$, we may calculate $P^i(\cdot)$ from Corollary 1, and using such program, by Corollary

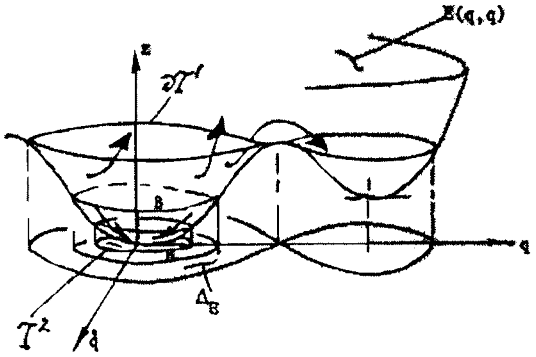

Fig. 5 Capturing subtargets enclosed in the energy well

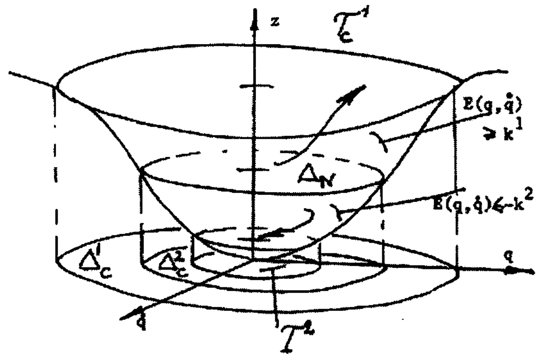

Fig. 6

2, we may push $v_0^i > 0$ as far from ∂T_c^i as in eq. (15) will hold, the distance measured in terms of the chosen $V^i(\bar{x})$. Player 2 aims at the bottom of the well (0, 0) enclosed by the small capturing target $\partial T_c^1 : E(q, \dot{q}) = c^2$ and chooses $V^2(\bar{x}) = E(q, \dot{q})$. Condition (iii) of Theorem 1 now reads: there is T_c^2 such that

$$\dot{E}(q, \dot{q}) \leqq -\frac{v_0^2 - c^2}{T_c^2} \tag{18}$$

Thus eq. (16) requires

$$\min_{u^2} \max_{u^1} \dot{E}(q, \dot{q}) \leqq \frac{v_0^2 - c^2}{T_c^2} \tag{19}$$

which is *the control condition for player 2* specifying $P^2(\cdot)$. For player 1 we start from below the threshold $\partial T_c^1 : E(q, \dot{q}) = c^1 = \frac{1}{4}\alpha^2/\beta$ and choose $V^1(\bar{x}) = c^2 - E(q, \dot{q})$ so that $V^1(\bar{x})$ at the threshold equals $c^2 - c^1$. Then eq. (15) becomes $\dot{V}^1(\bar{x}) = -\dot{V}^2(\bar{x}) = -\dot{E}(q, \dot{q}) \leqq -\frac{v_0^1 - c^1}{T_0^1}$, $v_0^1 < c^1$, and condition (iii) reads: there is T_c^1 and such that

$$\dot{E}(q, \dot{q}) \geqq \frac{c_1 - v_0^1}{T_c^1} \tag{20}$$

accumulating the energy i.e., pushing the motions of eq. (1) towards the threshold and above. Consequently eq. (16) becomes

$$\max_{u^1} \min_{u^2} \dot{E}(q, \dot{q}) \geqq \frac{c^1 - v_0^1}{T_c^1} \tag{21}$$

which is *the control condition for player 1* specifying $B^1(\cdot)$. Note that eq. (18) and eq. (20) are qualified by T_c^2, T_c^1 respectively and thus so is the choice of $P^i(\cdot)$. The rate of change of $V^i(\bar{x})$ is bounded by $k^i(T_c^i, v_0^i) = |v_0^i - c^i| / T_c^i$ and when Δ_0^i, T_c^i are given, so is k^i and $P^i(\cdot)$ can be exactly calculated. The latter happens when T_c^i and Δ_0^i are stipulated. Assume for a moment that they are not, as required by Definition 1 and the search for Δ_c^i.

Observe that with $P^2(\cdot)$ satisfying eq. (19) close to ∂T_c^2 we may lift $k^2 > 0$ and there is $k_*^2 = \sup[(v_0^2 - c^2)/T_c^2]$ for which $\dot{E}(q, \dot{q}) < 0$, i.e., for any $k^2 > k_*^2$ we have

$$\dot{E}(q, \dot{q}) \geqq 0 \tag{22}$$

contradicting (iii) expressed in terms of our chosen $V^2(\cdot)$. Note that condition (iii) is also necessary. Thus the contradicting eq. (22) defines complement to the maximal Δ_0^2 i.e., Δ_N^2, and since such complement is closed, it defines also an open Δ_c^2 (see Fig. 6).

Symmetrically, with $P^1(\cdot)$ satisfying eq. (21) close to the threshold, we may lower the rate $k^1 > 0$ and find $k_*^1 = \inf((c^1 - v_0^1)/T_c^1)$ for which $\dot{E}(q, \dot{q}) > 0$ i.e., for any $k^1 < k_*^1$ we have already

$$\dot{E}(q, \dot{q}) \leqq 0 \tag{23}$$

contradicting (iii) expressed in terms of our $V^1(\cdot)$. Again since (iii) is necessary, the contradicting eq. (23) defines the complement to maximal Δ_0^1 i.e., Δ_N^1 and thus Δ_c^1 (Skowronski 1989).

The first contradiction eq. (22) gives the candidate E-level for B^2, the second i.e., eq. (23) gives such candidate level for B^1. To check B^2

we choose $V_B^2(\bar{x}) = V^1(\bar{x}) = c^2 - E(q, \dot{q})$ with the same $P^2(\cdot)$ calculated from eq. (19) with $k^2 = k_*^2$ so that for eq. (22) Theorem 2 holds. Similarly to check B^1 we choose $V_B^1(\bar{x}) = V^2(\bar{x}) = E(q, \dot{q})$ with the same $P^1(\cdot)$ calculated from eq. (21) with $k^1 = k_*^1$. By eq. (23) Theorem 2 holds again. Thus the semibarriers B^i coincide with the found boundaries $\partial\Delta_i^c$, $i=1, 2$. It follows immediately that the points $(q, \dot{q}) \in \Delta$ where $\dot{E}(q, \dot{q}) = k$, $k_*^2 < k < k_*^1$ form the neutral set Δ^N. This set is a natural locus for the barrier B if it exists i.e., when the set specified by $\dot{E}(q, \dot{q}) = k$ forms a single E level, say an unstable cycle. But Δ^N of the above may also accommodate several surfaces Σ^i covering a number of unstable cycles and limit cycles, as the semibarriers B^1, B^2 may be located far apart with B empty.

Substituting now eq. (10) and eq. (4) into eq. (19) and eq. (21) we specify the control conditions obtaining for the player 2 :

$$\max_{u^2}(u^2\dot{q}) \geqq k_*^2 - d(q, \dot{q})\dot{q} + \max_{u^1}(u^1\dot{q}) \quad (24)$$

and for the player 1 :

$$\max_{\lambda}(\lambda\dot{q}\sin wt) \geqq k_*^1 + d(q, \dot{q})\dot{q} + \max_{u^2}(u^2\dot{q}) \quad (25)$$

as in any mechanical system, part of the control programs calculated from eqs. (24), (25) will be divided by $|\dot{q}|$ and blow up above saturation level for $\dot{q} \to 0$ i.e., when crossing the q-axis. Fortunately this axis is crossed instantaneously at all regular points of the phase space, so when the controller is switched off $u^1=0$ or left at the saturation level $u^1=\text{const}=\hat{u}^i$ suitably close to the q-axis say for $|\dot{q}| \leqq \beta^i$, the motion will carry on over the $\dot{q}=0$ point. The constants β^i may be calculated (Skowronski, 1989) but it has been proved more practical to pick them up by experience during the simulation process. The controllers that would satisfy eqs. (24), (25) are as follows.

$$u^2 \operatorname{sign}\dot{q} \begin{cases} \geqq \dfrac{k_*^2}{|\dot{q}|} - d(q, \dot{q})\operatorname{sign}\dot{q} + \hat{u}^1, \; |\dot{q}| \geqq \beta^2 \\ = \text{suitable constant}, \; |\dot{q}| < \beta^2 \end{cases} \quad (26)$$

and

$$\lambda \operatorname{sign}(q\sin\omega t) \begin{cases} \geqq \dfrac{k_*^1}{|\dot{q}\sin\omega t|} + \dfrac{d(q, \dot{q})\operatorname{sign}\dot{q}}{|\sin\omega t|} + \dfrac{\hat{u}^2}{\sin\omega t}, \\ |\dot{q}| \geqq \beta^2 \\ = \text{suitable constant}, \; |\dot{q}| < \beta^2 \end{cases} \quad (27)$$

In eq. (24) and in the program eq. (26) it is more convenient to use the known saturation value $\hat{u}^1$ rather than $\hat{\lambda}$. for the controller of player 1 we then have

$$u^1 \begin{cases} \geqq \dfrac{k_*^1}{|\dot{q}|} - d(q, \dot{q}) + \hat{u}^2\operatorname{sign}\dot{q} + \hat{u}^1, \; |\dot{q}| \geqq \beta^1 \\ = \text{suitable constant}, \; |\dot{q}| < \beta^1 \end{cases}$$

The above controllers secure the objectives of capture, but not necessarily the limiting behavior and chaos. That is why eqs. (26), (27) are left as inequalities, allowing the design of u^2, λ generating the other aims. They may be, in particular chosen as constant. This, in our case refers, particularly to the program for player 1 which is the active controller of the system. Since eq. (27) secures negative derivative $\dot{E}(q, \dot{q})$ up to reaching the threshold, from the basic Lyapunov stability theorem we learn that the threshold will be a stable attractor. Then ω, λ decide about their periodicity and/or chaos.

4. Example

To illustrate our argument we consider the subcase of eq. (1) with $d=0.1$, $\alpha=-1$, $\beta=1$, $\omega=1$, modeling a vibrating buckled beam :

$$\ddot{q} + (0.1 + u^2)\dot{q} - q + q^3 = \lambda\sin\omega t$$

The negative α establishes the basic equilibrium (origin (0, 0)) as an unstable saddle i.e., at the threshold between two symmetric neighboring wells for negative and positive q, respectively, centered at the Dirichlet stable equilibria $q^e=1$. A damped but unforced system $u^2\equiv0$, $u^1\equiv0$ has point attractors at these equilibria. For suitable values, say $u^2=0.15$, $\lambda=0.191$, we find a small amplitude limit cycle covered by T_c^2 and a larger amplitude chaotic motion covered by T_c^1 (Thompson, 1986). The latter is a steady state chaotic attractor which is almost large enough to cross the threshold, and some of its foldage may do so. The suitable values of u^2 and λ, if to be selected constant, may be discussed in the so called, "control space" $O\lambda u^2$ where for each pair λ, u^2 a specified phase space portrait of $Oq\dot{q}$ is prescribed. For small values of α, either negative or positive, the control space pattern is similar as eq. (28) may be shown structurally stable (Thompson, 1986). Then we can use the results of Ueda (1980), who studied it in great detail for the case $\alpha=0$. It follows that there are large regions in $O\lambda u^2$ for which computing limit cycles of our kind are separated by a single unstable cycle forming the barrier shown above.

In order to specify the barrier we let $c^2=0$ leading to $V^1(q, \dot{q})=-E(q, \dot{q})$, $V^2(q, \dot{q})=E(q, \dot{q})$ in eq. (16). Then the controls u^1, u^2 may be calculated from the condition that they should be the same for $\max_{u^1}\min_{u^2}$ and $\min_{u^2}\max_{u^1}\dot{E}(q, \dot{q})$ and $E=\frac{1}{2}\dot{q}^2-\frac{1}{2}q^2+\frac{1}{4}q^4$, we obtain

$$u^1=\begin{cases} u_{\max} & \text{if } \dot{q}\sin(\omega t)>0 \\ 0 & \text{if } \dot{q}\sin(\omega t)<0 \end{cases}$$

and

$$u^2=u^2_{\max}$$

The contour plot of energy function $E(q, \dot{q})$ for the above parameters at several levels is plotted in Fig. 7.

Lyapunov numbers spectrum in $u^1_{\max}$ parameter space, with $u^2_{\max}=0.2$, for two different starting points, $x(t_0)=(1.1, 0.0)$ and $(2.5, 1.0)$-one is near the bottom of the energy well and the other is away from it, is plotted in Fig. 8. One can notice that there are two different values in the plot for a $u^1_{\max}$ since it was not calculated

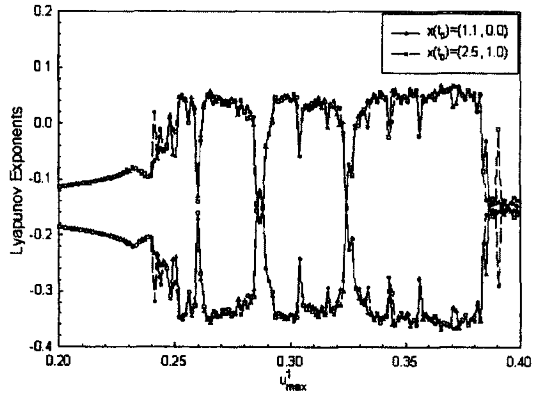

Fig. 8 Lyapunov exponents in $u^1_{\max}$ parameter space with two different initial conditions

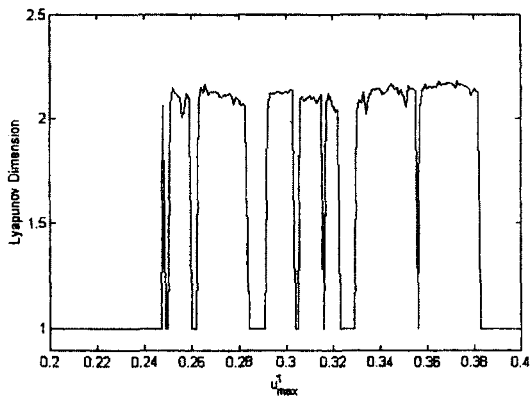

Fig. 9 Lyapunov dimension in $u^1_{\max}$ parameter space

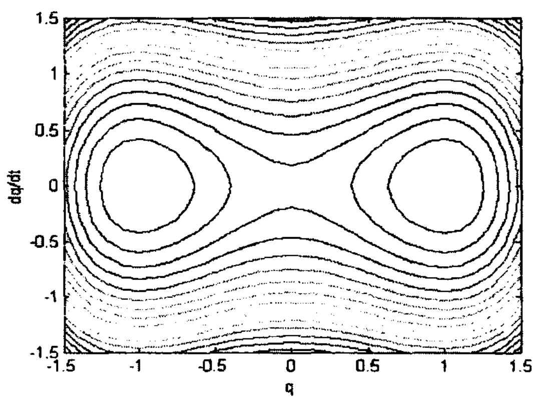

Fig. 7 Contour plot of an energy function

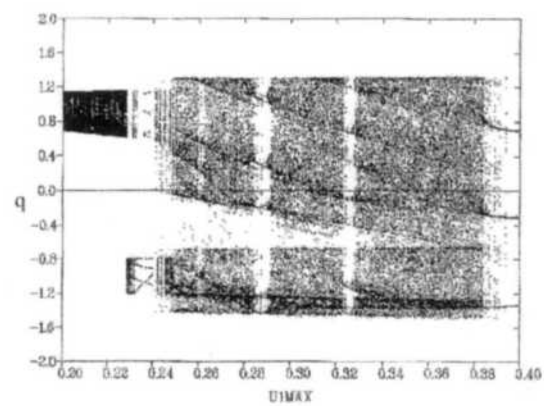

Fig. 10 Bifurcation diagram with $x(t_0)=(1.1, 0.0)$

long enough for convergence. Also Lyapunov dimension in u^1_{max} parameter space is plotted in Fig. 9. In Figs. 10 and 11, the Poincare maps projected on q axis in u^1_{max} space for two different starting points are plotted.

From Lyapunov numbers spectrum, Lyapunov dimension plot, and Poincaré section plot, one may identify the characteristics of the trajectories. We chose four u^1_{max} by inspecting those plots; 0.237, 0.241334, 0.24975, and 0.255 which represent limit cycles inside the two energy wells, a quasi-periodic limit cycle, a large limit cycle jumping around the two energy wells, and a chaotic attractor, respectively. These trajectories and cor responding Poincaré maps are in Figs. 12, 13, 14, 15, 16, and 17. The typical u^1 and excitation for

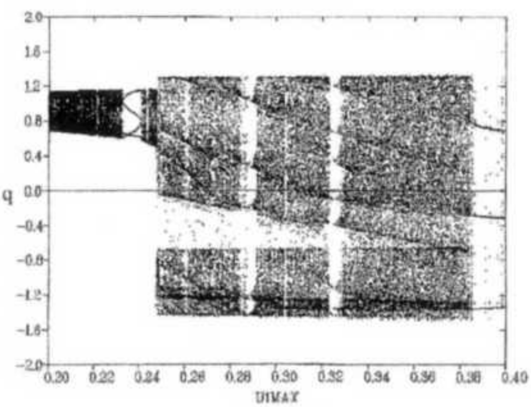

Fig. 11 Bifurcation diagram with $x(t_0) = (2.5, 1.0)$

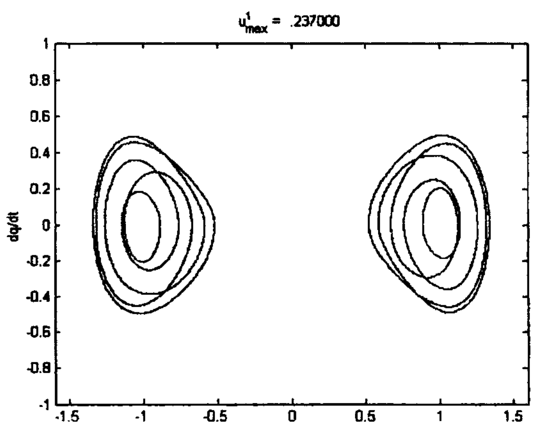

Fig. 12 Limit cycles inside the energy wells ($u^1_{max} =$ 0.237)

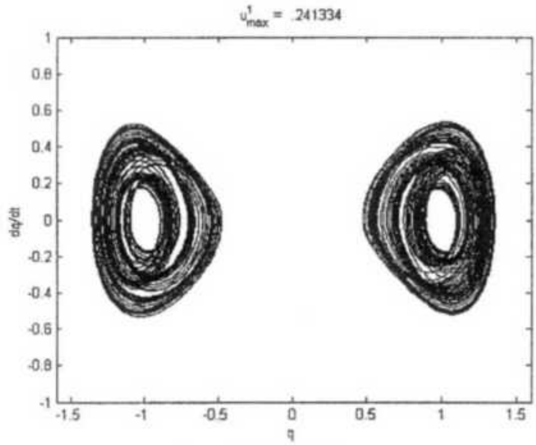

Fig. 13 Quasi-periodic limit cycles ($u^1_{max} =$ 0.241334)

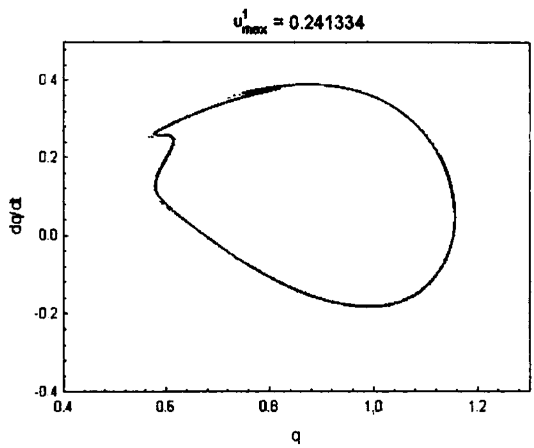

Fig. 14 Poincaré map for a quasi-periodic limit cycles as in Fig. 13

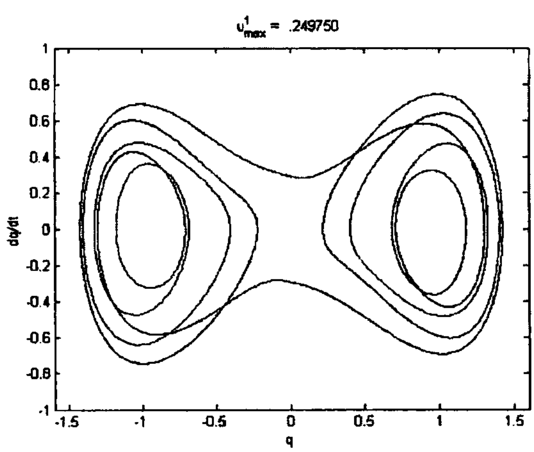

Fig. 15 A large limit cycle (u^1_{max}=0.24975)

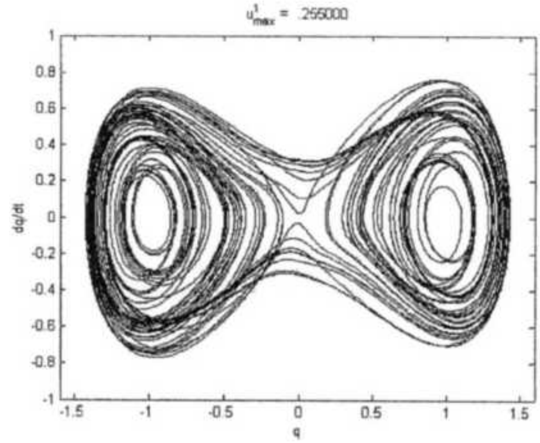

Fig. 16 A chaotic attractor (u^1_{max}=0.255)

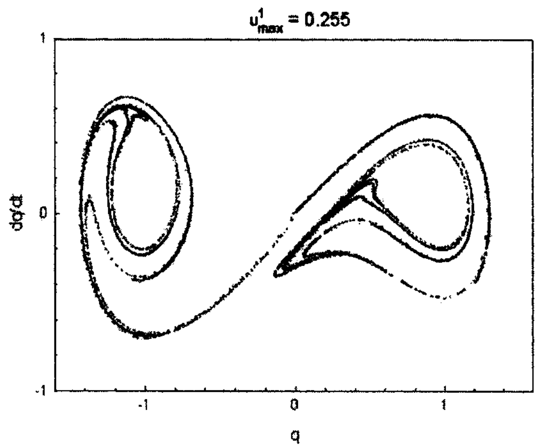

Fig. 17 Poincaré map for a chaotic attractor as in Fig. 16

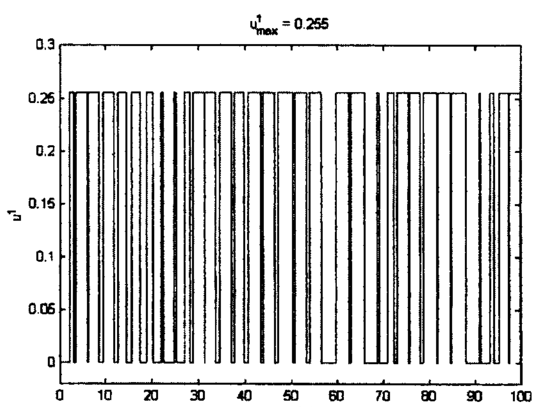

Fig. 18 u^1's time history generating a chaotic attractor (u^1_{max}=0.255)

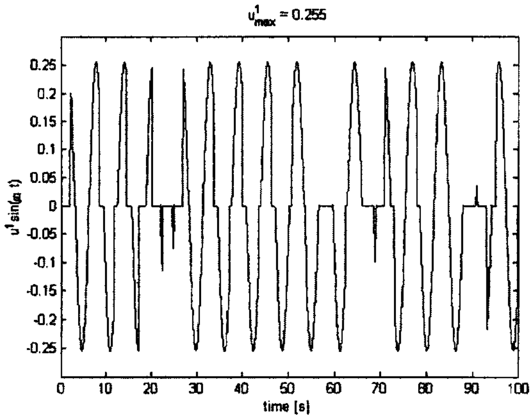

Fig. 19 Excitation time history generating a chaos (u^1_{max}=0.255)

u^1_{max}=0.255 are plotted in Figs. 18 and 19, respectively. For the case u^1_{max}=0.24975, the authors' premise that if there is a chance of pushing the motions above the threshold, out of the well, then the controller can make the system chaotic does not hold.

5. Summary

Feedback Strategies of a qualitative competitive game between two players has been designed such as to influence parameters of a mechanical system to induce chaotic behavior. The purpose is to reduce the options and effects of the opponent's strategy. We show it on a case with dynamics specified by a nonautonomous Duffing equation with the players represented by damping and external forcing, respectively, as an example. However, the authors' premise that if there is a chance of pushing the motions above the threshold, out of the well, then the controller can make the system chaotic is not satisfied for some cases. There are several regions in parameter space where this premise does not hold.

References

Arderna, M. D., Skowronski, J. M., 1989, "Coordination Controllers for Multi-arm Manipulators — A Case Study", *Adv. Control & Dyn. Systems*, Acad. Press.

Awrejcewicz, J., 1988, "Chaotic Motion in a Nonlinear Oscillator with Friction," *KSME International Journal*, Vol. 2, No. 2.

Filippov, A. F., 1977, "Existence of Solutions of Generalized Differential Equations," *Math Notes*, Vol. 10, pp. 1267~1272.

Lee, B. S., 1995, "Chaos Maximizing Optimal Control," *KSME International Journal*, Vol. 9, No. 4.

Skowronski, J. M., 1986, *Control dynamics of robotic manipulators*, Academic Press.

Skowronski, J. M. and Stonier, R. J., 1987, "The Barrier in the Pursuit-Evasion Game with Two Targets," *Int. J. Computers & Math. Appl.*, Vol. 13, pp. 37~45.

Skowronski, J. M. and Vincent, T. L., 1988, "Playability with and Without Capture," *J. Opt. Th. Appl.*, Vol. 36.

Skowronski, J. M., 1989, *Control of nonlinear mechanical systems*, Plenum, NY.

Thompson, J. M. T. and Stewart, H. B., 1986, *Nonlinear dynamics and chaos*, Wiley.

Ueda, V., 1980, "Explosion of Strange Attractors Exhibited by Duffing Equation," in R. H. Helleman (ed), *Nonlinear Dynamics*, N. Y. Ac. Sci., pp. 422~434.